compact scattered spaces as attractors of generalized iterated function systems

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Hutchinson-Barnsley theorem

Theorem (Hutchinson, Barnsley, 1980's)
If X is a complete metric space and $\mathcal{F}$ is a finite family of Banach contractions of X ($\mathrm{Lip}(f) < 1$ for $f \in \mathcal{F}$), then there exists a unique nonempty and compact set $A_{\mathcal{F}} \subset X$ such that

$$A_{\mathcal{F}} = \bigcup_{f \in \mathcal{F}} f(A_{\mathcal{F}}).$$

Remark
It is enough to assume that each $f \in \mathcal{F}$ is *weak* contraction (in the sense of Rakotch, Browder, Matkowski). If X is compact, then it means

$$d(f(x), f(y)) < d(x, y), \quad x, y \in X, \ x \neq y.$$

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IFS fractals

Definition

(*) A finite family $\mathcal{F}$ of weak [Banach] contractions will be called *a weak [Banach] iterated function system (IFS)*.

(*) A set $A_{\mathcal{F}}$ which satisfy the thesis of the H-B theorem will be called *an attractor* or *fractal generated by $\mathcal{F}$*.

(*) A compact metric space X is called *weak [Banach] IFS fractal*, if it is an attractor of some weak [Banach] IFS.

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generalized IFS

If X is a metric space and $m \in \mathbb{N}$, then we endow the Cartesian product X^m with the maximum metric d_m.

Definiton

(*) A map $f : X^m \to X$ is called *a generalized Banach contraction of order m*, if $\mathrm{Lip}(f) < 1$.

(*) A map $f : X^m \to X$ is called *a generalized weak contraction of order m*, if ... it satisfies weaker contractive condition, which in the case when X is compact reduces to

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problem - the class of GIFSs' fractals

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Is the class of GIFSs' fractals essentially wider than the class of IFSs' fractals?

Which sets/spaces are GIFSs' fractals?

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examples

Example (Mihail, Miculescu 2010)
The Hilbert cube $X = [0,1] \times \left[0, \frac{1}{2}\right] \times \left[0, \frac{1}{4}\right] \times ...$:
($*$) is a Banach GIFS fractal of order 2, for a GIFS $\mathcal{G} = \{f, g\}$, where

$$f((x_k),(y_k)) = \frac{1}{2}(x_1, y_1, y_2, ...) \quad g((x_k),(y_k)) = \frac{1}{2}(1 + x_1, y_1, y_2, ...);$$

($*$) is not a Banach IFS fractal (as it has infinite dimension);
However, it is not known whether it a weak IFS fractal.
Example (S. 2013)
(1) For each $m \geqslant 2$, there exists a Cantor set $C(m) \subset \mathbb{R}^2$ such that:
($*$) $C(m)$ is a Banach GIFS fractal of order m;
($*$) $C(m)$ is not a weak GIFS fractal of order $m - 1$;
(2) There exists a Cantor set $C \subset \mathbb{R}^2$ which is not a weak GIFS fractal.
However, $C(m)$ is homeomorphic to the Cantor ternary set, the attractor of a Banach IFS on $\mathbb{R}$.

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scattered spaces

A topological space X is called *scattered*, if every its nonempty subspace has an isolated point.

Fact Compact metrizable topological space is scattered iff it is countable.

For a scattered space X, define the Cantor-Bendixon derivative by

$$X' := \{x \in X : x \text{ is an accumulation point of } X\}$$

For each ordinal α, define the Cantor-Bendixon α-th derivative $X^{(\alpha)}$ by
$(*)$ $X^{(\alpha+1)} := (X^{(\alpha)})'$;
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The *scattered height* of X is defined by $\text{ht}(X) := \min\{\alpha : X^{(\alpha)} \text{ is finite}\}$.

Theorem (Mazurkiewicz-Sierpiński)
Each metrizable compact scattered space X is homeomorphic to the space $\omega^\beta \cdot n + 1$ for some $\beta < \omega_1$, and in this case $\text{ht}(X) = \beta$ and $\text{card}(X^{(\text{ht}(X))}) = n$.

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Theorem (Nowak, 2013)
Let X be a metrizable compact scattered space.

(1) If ht(X) is a successor ordinal, then X is homeomorphic to a set $A \subset \mathbb{R}$ which is a Banach IFS fractal.
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(1) If $\mathrm{ht}(X)$ is a successor ordinal, then X is homeomorphic to a set $A \subset \mathbb{R}$ which is a Banach IFS fractal.

(2) If $\mathrm{ht}(X)$ is limit ordinal, then X is not homeomorphic to any weak IFS fractal.

(3) X is homeomorphic to a subset $A \subset \mathbb{R}$ which is not a weak IFS-fractal.

Theorem (Maślanka, S. 2017)

Let X be a metrizable compact scattered space.

(1) X is homeomorphic to a set $A \subset \mathbb{R}$ which is a Banach GIFS fractal of order 2.

(2) For every $m \in \mathbb{N}$, X is homeomorphic to the set $A \subset \mathbb{R}$ such that:
   - (*) A is a Banach GIFS fractal of order m;
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proof - certain scattered subsets of real line

Construction of sets L_α (Nowak, 2013)
Fix a limit ordinal $\delta_0 < \omega_1$. For every $\alpha \leqslant \delta_0$, there exists a sequence (α_n) such that

(a) for every $\alpha \leqslant \delta_0$, the sequence $(\alpha_n + 1) \nearrow \alpha$;

(b) for every $\alpha \leqslant \beta \leqslant \delta_0$, we have $\alpha_n \leqslant \beta_n$ for every $n \in \mathbb{N}$.

For $n \in \mathbb{N}$, let $s_n(x) := r^n x + r^n$, where $r < 1$. Then define the family L_α, $\alpha \leqslant \delta_0$ in the following inductive way:

(1) $L_0 := \{0\}$;

(2) $L_\alpha = L_0 \cup \bigcup_{n=1}^{\infty} s_n(L_{\alpha_n})$.

Fact
L_α is scattered space with $\mathrm{ht}(L_\alpha) = \alpha$ and $(L_\alpha)^{(\alpha)} = \{0\}$. In particular, L_α is homeomorphic to $\omega^\alpha + 1$.

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proof of (1) - unital case

Lemma (Nowak, 2013)
For every $\alpha \leqslant \delta_0$, there exists a map $g_\alpha : [0,1] \to [0,1]$ such that

(i) $\mathrm{Lip}(g_\alpha) \leqslant \frac{1}{1-2r}$;

(ii) if $\alpha \leqslant \beta \leqslant \delta_0$, then $g_\alpha(L_\beta) = L_\alpha$.

Given $\alpha < \omega_1$ define the map $F, G : L_\alpha \times L_\alpha \to L_\alpha$ by

$$G(x,y) := s_1(g_{\alpha_1}(x)) \quad \text{and} \quad F(x,y) := \begin{cases} s_{n+1}(g_{\alpha_{n+1}}(x)) & \text{if} \quad y \in s_n(L_{\alpha_n}) \\ 0 & \text{if} \quad y = 0 \end{cases}$$

Then:

$$\mathrm{Lip}(F) \leqslant \frac{2r}{1-2r} \qquad \mathrm{Lip}(G) \leqslant \frac{r}{1-2r}$$

$$F(L_\alpha \times L_\alpha) \cup G(L_\alpha \times L_\alpha) = \left(L_0 \cup \bigcup_{n=1}^{\infty} s_{n+1}(L_{\alpha_{n+1}}) \right) \cup (s_1(L_{\alpha_1})) = L_\alpha$$

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proof of (1) - the general case

Theorem (Maślanka, S. 2017)
Let X be a metrizable compact scattered space.

(1) **X is homeomorphic to a set $A \subset \mathbb{R}$ which is a Banach GIFS-fractal of order 2.**

(2) For every $m \in \mathbb{N}$, X is homeomorphic to the set $A \subset \mathbb{R}$ such that:
   (*) A is a Banach GIFS fractal of order m;
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(3) X is homeomorphic to a set $A \subset \mathbb{R}$ which is not a weak GIFS-fractal.

Fact
If X is a metric space of the form $X = X_1 \cup ... \cup X_n$, where
(i) each X_i is a Banach GIFS fractal of order 2;
(ii) $\min_{i \neq j} \operatorname{dist}(X_i, X_j) > \max_i \operatorname{diam}(X_i)$,
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proofs of (2) and (3)

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Modify sets L_α in the following way:

- (*) replace s_n by another transformation s_n^α (different at each level α);
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next results - further modifications

Theorem(Maślanka, S., 2017)
Let Z be a connected Banach GIFS fractal of order m. There exists a compact metric space X such that:

(1) each connected component of X is a homothetic copy of Z;
(2) X is not homeomorphic to a weak IFS fractal;
(3) X is a Banach GIFS fractal of order $\max\{2, m\}$.

Proof(sketch)
Replace each point in L_ω by appropriately small copy of Z.

Corollary
For every $n \in \mathbb{N}$ and real $1 \leqslant s \leqslant n$, there exists a set $A \subset \mathbb{R}^n$ such that:

(i) $\dim_H(A) = s$;
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open problems

Problems

(1) Let $m \geqslant 2$. Is there a compact metric space X which is a Banach GIFS fractal of order m, but which is not homeomorphic to a weak GIFS fractal of order $m-1$?

(2) Does there exists a Peano continuum X which is a Banach GIFS fractal, but which is not (homeomorphic to) weak IFS fractal?

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acknowledgments

Thank You For Your Attention!

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